Classifying spaces of monoids

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Stable homology through scanning
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Examples:

M = Self-Homotopy-Equivalences (S^n) — BM classifies spherical fibrations

M = Self-Homotopy-Equivalences () — $BM = B\mathrm{Out}(F_n)$

Just as for groups, if M is a topological monoid we define

$$BM = \coprod_{n \geq 0} M \times \cdots \times M \times \Delta^n \,/\, \text{identifications}$$

where the faces of $[m_1|\cdots|m_n]$ are $[m_2|\cdots|m_n], [m_1 m_2|\cdots|m_n], \ldots, [m_1|\cdots|m_{n-1} m_n], [m_1|\cdots|m_{n-1}$

That is:

$$\begin{array}{cccc} [& m_1 | & \cdots | & m_n] \\ \uparrow & \uparrow & \uparrow & \uparrow \\ B & A & A & B \end{array}$$

delete one of the $n+1$ walls;
A) If two elements come together, concatenate them.
B) If we remove an outer wall, the orphaned element falls off the edge.

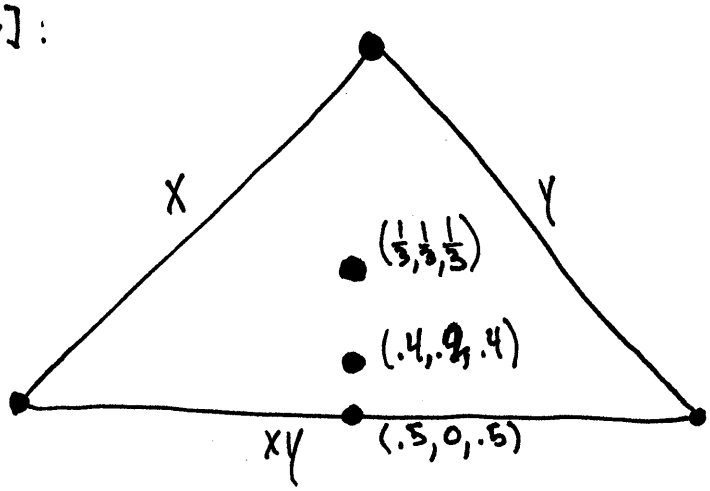
Barycentric coordinates on Δ^n: $(t_0, t_1, \ldots, t_n)$ w/ $t_i \geq 0$ and $t_0 + t_1 + \cdots + t_n = 1$

We can think of the t_i as weights on the walls:

$$\begin{array}{cccc} [& m_1 | & \cdots | & m_n] \\ \uparrow & \uparrow & \uparrow & \uparrow \\ t_0 & t_1 & t_{n-1} & t_n \end{array}$$

When a weight goes to 0, we delete that wall.

$[x|y]$:

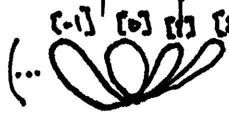


Understanding BM

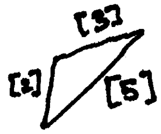
Warm-up: what is $B\mathbb{N}$?

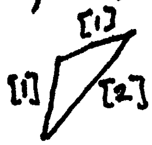
First, understand $B\mathbb{Z}$ using this definition.

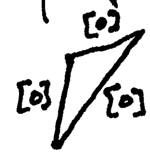
(We'll focus on π_1, which only depends on the 2-skeleton.)

$\pi_1(\text{1-skeleton of } B\mathbb{Z}) = \pi_1($ 

$) = \text{free group } \langle \dots, a_{-1}, a_0, a_1, a_2, \dots \rangle$

2-cells have the form 

or 

or 

$$\pi_1(B\mathbb{Z}) = \left\langle \dots, a_{-1}, a_0, a_1, a_2, \dots \;\middle|\; \begin{matrix} a_2 a_3 = a_5 \\ a_1 a_1 = a_2 \\ a_0 a_0 = a_0 \end{matrix} \right\rangle = \langle a_1 \rangle = \mathbb{Z}$$

In general for a discrete group G,

$\pi_1(BG) = \langle \text{elements of } G \mid \text{multiplication table of } G \rangle = G$

$\pi_1(\text{1-skeleton of } B\mathbb{N}) = \pi_1($ 

$) = \langle a_0, a_1, a_2, \dots \rangle$

$$\pi_1(B\mathbb{N}) = \left\langle a_0, a_1, a_2, \dots \;\middle|\; \begin{matrix} a_2 a_3 = a_5 \\ a_1 a_1 = a_2 \\ a_0 a_0 = a_0 \end{matrix} \right\rangle = \langle a_1 \rangle = \mathbb{Z}$$

In general for a discrete monoid M,

$\pi_1(BM) = \langle \text{elements of } M \mid \text{multiplication table of } M \rangle = \text{group-completion of } M$

We saw last time that $G \simeq \Omega BG$ for any topological group G.

For monoids, $M \simeq \Omega BM$ if M is connected (or even if $\pi_0(M)$ is a group) but not in general.

Group Completion Theorem:

$M \to \Omega BM$ induces $H_*(M)[\pi_0^{-1}] \approx H_*(\Omega BM)$

where the left side $H_*(M)[\pi_0^{-1}]$ is $H_*(M)$ with the action of π_0 forcibly inverted.

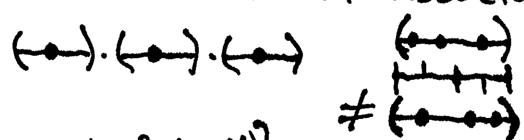
We need certain conditions to apply the theorem (or even for $H_*(M)[\pi_0^{-1}]$ to make sense);

the condition we will use in this course is that the multiplication in M is commutative-up-to-homotopy.

Goal: prove that B(points in an interval)
$= $ points in $\mathbb{R}$ that can disappear at ∞

This is the first example of the <u>relaxation principle</u>:
the principle that if we start with the space of some objects in a bounded interval we can pass to the classifying space / find a delooping by relaxing and letting them vanish at ∞.

We want to work with the space $\{$finite subsets of $(0,1)\}$
under juxtaposition and rescaling: 
But this is not a monoid because it's not associative:



So let $M = \{$pairs $(\ell > 0$, finite subset of $(0,\ell))\}$
topologized so that ℓ can vary continuously,
and points can move around, but not disappear or coincide,
with operation given by juxtaposition <u>without</u> rescaling. (has no identity, but that's not a problem)

Let $Y = \{$finite subsets of $\mathbb{R}\}$
topologized so that points can disappear at $\pm\infty$
<u>Ex</u>. The sequence $y_n = \{1,2,3,n\}$ converges to $\{1,2,3\}$ (as does $y_n = \{1,2,3,\log n, n^2\}$).

Goal: prove $Y \simeq BM$. (relaxation principle for points in an interval)

Remark: there is no need to do this.
The homomorphism $M \longrightarrow \mathbb{N}$ is a homotopy equivalence,
finite set in $(0,\ell) \longmapsto$ its cardinality
so $BM \simeq B\mathbb{N} \simeq S^1$.

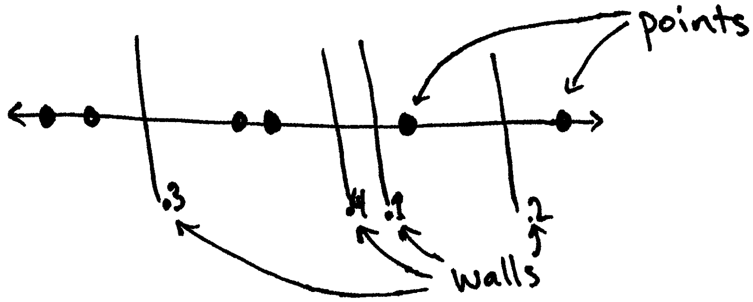
each component of M (e.g. finite sets with 7 elt's) is contractible — take straight-line homotopy to 7 equally-spaced points in $(0,1)$

But we want a proof that doesn't require knowing the topology on M.
(In general, the purpose of applying the relaxation principle will be to understand the topology of M.)

Proof of relaxation principle

We interpolate between Y and BM using a new space S.

$$Y \xleftarrow{\simeq} S \xrightarrow{\simeq} BM$$



A point in S is:

- a finite collection of points in $\mathbb{R}$,
- a finite collection of $n+1 \geq 1$ "walls" disjoint from the points, and
- weights t_i on the walls with $t_i > 0$ and $t_0 + \cdots + t_n = 1$.

S is topologized so that:

- when the weight on a wall goes to 0, that wall disappears
- points cannot pass through each other or through walls (unless the weight goes to 0 first.)
- points can disappear at $\pm\infty$, but walls cannot
- adjacent walls cut the real line into "slabs"; these slabs need not contain any points, but must have positive thickness

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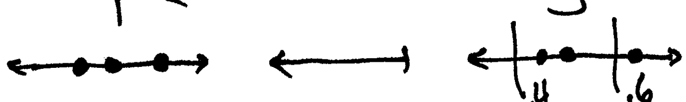
$$Y \xleftarrow{\text{forget walls}} S$$



Why is this map $S \to Y$ a homotopy equivalence?

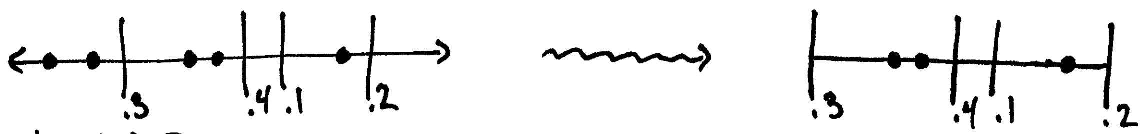
1) It is a fibration.
2) The fibers are contractible.

Proof of 2) The fiber over some point consists of all ways we can insert walls: i.e. all <u>convex combinations</u> of points in the complement.
$\Sigma t_i p_i$ with $0 \leq t_i$, $\Sigma t_i = 1$
So the fiber is <u>convex</u> and thus contractible.

Proof of 1) We can check that the map is a fibration (satisfies the homotopy lifting property) using maps from compact sets.

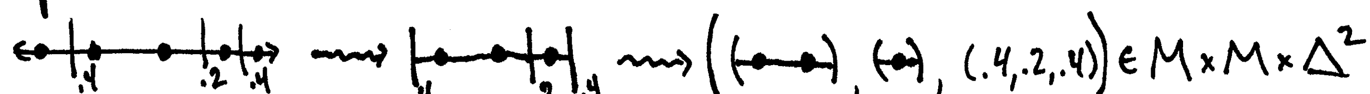
$S \longrightarrow BM$

The map $S \to BM$ is defined as follows:
First, forget whatever is to the left of the leftmost wall and to the right of the rightmost wall.

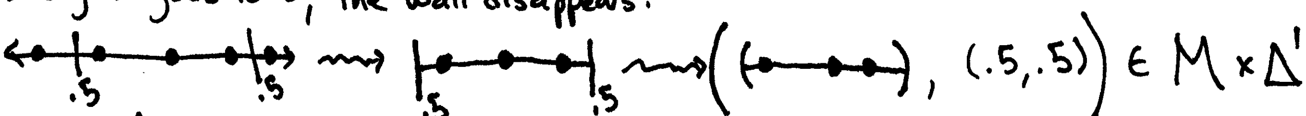


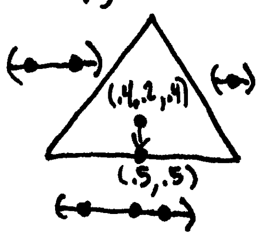
What's left?
We have $n+1$ walls cutting up into n slabs: , ,
together with $n+1$ weights t_i with $t_i > 0$ and $t_0 + \cdots + t_n = 1$.
Each slab is an open interval (of some positive length) with some points in it; that is, each slab is an element of M, so we obtain an element of $\underbrace{M \times \cdots \times M}_{n \text{ times}}$.
Taking the weights as barycentric coordinates on Δ^n, we get a point in $M \times \cdots \times M \times \Delta^n$, which sits inside BM by construction.

Example:



As a weight goes to 0, the wall disappears:





As the points move, the slabs move continuously in the $M \times \cdots \times M$ factor.

Why is this map $S \to BM$ a homotopy equivalence?
1) It is a fibration. (Proof similar to previous argument.)
2) The fibers are contractible.

Example for 2)
The fiber over $(\ , \ , (.4,.2,.4)) \in BM$ is the product of:
- what could have been to the left of the leftmost wall (contractible by sliding to $-\infty$)
- what could have been to the right of the rightmost wall (contractible by sliding to $+\infty$)
- a translational factor because we only remembered *relative* separation of walls, not absolute position (this factor is just $\mathbb{R}$)

This concludes the proof: $Y \simeq S \simeq BM$, as desired.